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On the functoriality of cohomology of categories[☆]

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Abstract

In this paper we show that the Baues–Wirsching complex used to define cohomology of categories is a 2-functor from a certain 2-category of natural systems of abelian groups to the 2-category of chain complexes, chain homomorphisms and relative homotopy classes of chain homotopies. As a consequence we derive (co)localization theorems for this cohomology.

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1. Introduction

Baues–Wirsching cohomology of a small category \mathbf{C} with coefficients in a natural system L on \mathbf{C} was defined in [2] as the cohomology of a certain cochain complex $F^*(\mathbf{C}, L)$. This cohomology generalizes some other cohomologies previously known, as for example:

- the Hochschild–Mitchell cohomology of \mathbf{C} with coefficients in a functor $L: \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Ab}$ [10],
- the cohomology of the classifying space $B\mathbf{C}$ with local coefficients L ,
- the Mac Lane cohomology of a ring [8,7].

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The Baues–Wirsching complex $F^*(\mathbf{C}, L)$ as well as its cohomology $H^*(\mathbf{C}, L)$ are known to be functors on a certain category \mathbf{Nat} of pairs (\mathbf{C}, L) . It is also known that equivalences of categories induce homotopy equivalences in Baues–Wirsching complexes and isomorphisms in cohomology groups. However, this does not follow immediately from the fact that F^* and H^* are functors in \mathbf{Nat} . More precisely, the behaviour of $F^*(\mathbf{C}, L)$ and $H^*(\mathbf{C}, L)$ with respect to natural transformations between functors in the first variable is not known. The goal of this paper is to shed some light on that issue.

We define a new 2-category $\mathbf{Nat}_{\mathcal{F}}$ of pairs (\mathbf{C}, L) containing \mathbf{Nat} and prove that F^* is in fact a 2-functor in $\mathbf{Nat}_{\mathcal{F}}$ (Theorem 4.1). The 2-morphisms in the category of cochain complexes will be homotopy classes of homotopies relative to the boundary of the cylinder. The category obtained from $\mathbf{Nat}_{\mathcal{F}}$ by taking sets of connected components on morphism categories turns out to be a quotient of \mathbf{Nat} (Proposition 4.2) and the cohomology functor H^* factors through this quotient category (Corollary 4.3).

As an application of these results we obtain localization and colocalization theorems for Baues–Wirsching cohomology (Theorems 5.4 and 5.10). We also give some examples of how these (co)localization theorems can be used to carry out computations in cohomology of categories.

We believe that it is possible to dualize the results of this paper to the homology of small categories introduced by Pirashvili and Waldhausen [12]. This homology theory extends Mac Lane homology of rings [8], which is isomorphic to topological Hochschild homology in the sense of [3].

1.1. Notation and conventions

Compelled by the necessarily intricate notation of this paper, we have decided to include this paragraph to fix the meaning of some symbols. Additional notation will also appear in the development of the paper, but it will not contradict in any case that introduced here.

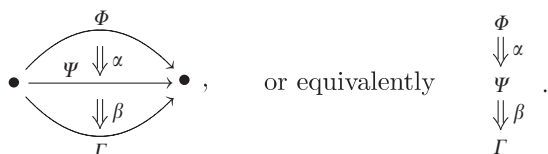
Boldface capital letters \mathbf{C} , \mathbf{D} and \mathbf{E} denote small categories; X and Y are objects; and f , g , h , k and σ are morphisms in these categories. Capital Greek letters Φ , Ψ , Γ and Υ stand for functors between those categories; and α , β , γ , ε , t and s for natural transformations. The arrow \rightarrow is used for morphisms in ordinary categories, 1-morphisms in 2-categories, functors, and 2-functors while \Rightarrow is kept for 2-morphisms and natural transformations. Identity morphisms or 2-morphisms are denoted by 1 . A subscript will clarify the meaning in ambiguous cases. The category of abelian groups is \mathbf{Ab} . The category of small categories and functors is \mathbf{Cat} , and the standard 2-category of small categories, functors and natural transformations is denoted by \mathbf{Cat}_2 . Natural systems, in the sense of Definition 3.1, are denoted by L , M and N . An arbitrary cochain in cohomology of categories is usually denoted by c . The letters A^* , B^* and C^* with superscript $*$ stand for cochain complexes of abelian groups. The differential of all cochain complexes is denoted by d . The letters p , q , h and r stand for graded morphisms between graded abelian groups.

These symbols can be altered by adding superscripts or subscripts.

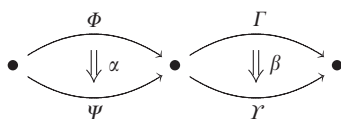
In 2-categories the word “morphism” will be synonym of “1-morphism”. All categories can be regarded as 2-categories with only the trivial 2-morphisms.

The symbol \bullet stands for an unspecified object in an arbitrary category. It can appear several times in a diagram, however in general it will stand for a different object each time.

The composition of morphisms, say $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, or functors will be indicated by juxtaposition gf . We use the symbol \square for the vertical composition of 2-morphisms or natural transformations $\beta \square \alpha: \Phi \Rightarrow \Gamma$ as in the diagram



Horizontal composition of 2-morphisms or natural transformations as in the following diagram



is also denoted by juxtaposition $\beta\alpha: \Gamma\Phi \Rightarrow \Upsilon\Psi$.

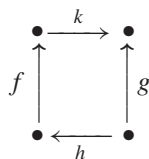
2. Factorization categories

Factorization categories are the source of the coefficient objects for the Baues–Wirsching cohomology of small categories (see Definition 3.1). A thorough study of their properties is essential to study in depth the functoriality of the Baues–Wirsching complex in Section 4.

Definition 2.1. The *factorization category* \mathcal{FC} of a small category \mathbf{C} has

objects: morphisms in \mathbf{C} ,

morphisms: $(h, k): f \rightarrow g$ are pairs of morphisms in \mathbf{C} such that $kfh = g$, that is commutative diagrams in \mathbf{C}



and composition is defined by $(h', k')(h, k) = (hh', k'k)$.

One can easily check that factorization categories define a functor

$$\mathcal{F}: \mathbf{Cat} \longrightarrow \mathbf{Cat}.$$

This functor is defined on morphisms as follows: a functor $\Phi: \mathbf{C} \rightarrow \mathbf{D}$ is sent to another one

$$\mathcal{F}(\Phi): \mathcal{FC} \longrightarrow \mathcal{FD}$$

which is given

on objects: by $\mathcal{F}(\Phi)(f) = \Phi(f)$,
 on morphisms: by $\mathcal{F}(\Phi)(h, k) = (\Phi(h), \Phi(k))$.

Notice that the functor \mathcal{F} preserves products. Therefore we can consider the 2-category $\mathbf{Cat}_{\mathcal{F}}$ obtained from \mathbf{Cat}_2 by applying the functor \mathcal{F} to morphism categories. Let us make explicit the structure of $\mathbf{Cat}_{\mathcal{F}}$:

(2.A) *objects*: are small categories;

1-morphisms: $\alpha: \mathbf{C} \rightarrow \mathbf{D}$ are actually natural transformations $\alpha: \Phi \Rightarrow \Psi$ between functors $\Phi, \Psi: \mathbf{C} \rightarrow \mathbf{D}$, and composition $\beta\alpha$ in $\mathbf{Cat}_{\mathcal{F}}$ is horizontal composition $\beta\alpha$ of natural transformations;

2-morphisms: $(\varepsilon, \gamma): \alpha \Rightarrow \beta$ are natural transformations such that $\gamma \square \alpha \square \varepsilon = \beta$, i.e., commutative diagrams of natural transformations

$$\begin{array}{ccc} \Psi & \xRightarrow{\gamma} & \Gamma \\ \alpha \uparrow & & \uparrow \beta \\ \Phi & \xleftarrow{\varepsilon} & \Gamma \end{array}$$

vertical composition of 2-morphisms is given by $(\varepsilon', \gamma') \square (\varepsilon, \gamma) = (\varepsilon \square \varepsilon', \gamma' \square \gamma)$, and the horizontal composition of (ε, γ) and (ε', γ') as in the following diagram in $\mathbf{Cat}_{\mathcal{F}}$

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{\alpha} & \mathbf{D} & \xrightarrow{\alpha'} & \mathbf{E} \\ & \Downarrow (\varepsilon, \gamma) & & \Downarrow (\varepsilon', \gamma') & \\ \mathbf{C} & \xrightarrow{\beta} & \mathbf{D} & \xrightarrow{\beta'} & \mathbf{E} \end{array}$$

is $(\varepsilon', \gamma')(\varepsilon, \gamma) = (\varepsilon' \varepsilon, \gamma' \gamma)$.

There is a unique 2-functor

$$\mathbf{Cat} \longrightarrow \mathbf{Cat}_2$$

which is the identity on objects and 1-morphisms. Moreover, there is also a unique 2-functor

$$\mathbf{Cat} \longrightarrow \mathbf{Cat}_{\mathcal{F}}$$

which is the identity on objects and sends a functor $\Phi: \mathbf{C} \rightarrow \mathbf{D}$ to the identity natural transformation $1_{\Phi}: \Phi \Rightarrow \Phi$ regarded as a morphism $1_{\Phi}: \mathbf{C} \rightarrow \mathbf{D}$ in $\mathbf{Cat}_{\mathcal{F}}$.

Proposition 2.2. *There is defined a 2-functor $\mathcal{F}: \mathbf{Cat}_{\mathcal{F}} \rightarrow \mathbf{Cat}_2$ fitting into a commutative diagram*

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{\mathcal{F}} & \mathbf{Cat} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\mathcal{F}} & \xrightarrow{\mathcal{F}} & \mathbf{Cat}_2 \end{array}$$

where the vertical arrows are the 2-functors previously defined.

Proof. The new 2-functor \mathcal{F} is defined in the following way. We use the notation in Definition 2.1 and (2.A):

on objects: $\mathcal{F}\mathbf{C}$ is the factorization category,

on 1-morphisms: the functor $\mathcal{F}(\alpha): \mathcal{F}\mathbf{C} \rightarrow \mathcal{F}\mathbf{D}$ is defined

on objects: given an object f in $\mathcal{F}\mathbf{C}$, which is a morphism $f: X \rightarrow Y$ in \mathbf{C} ,

$\mathcal{F}(\alpha)(f) = \alpha_Y \Phi(f) = \Psi(f) \alpha_X$;

on morphisms: $\mathcal{F}(\alpha)(h, k) = (\Phi(h), \Psi(k))$.

on 2-morphisms: $\mathcal{F}(\varepsilon, \gamma): \mathcal{F}(\alpha) \Rightarrow \mathcal{F}(\beta)$ is the natural transformation

which evaluated on f as above is the morphism $\mathcal{F}(\varepsilon, \gamma)_f = (\varepsilon_X, \gamma_Y)$ in $\mathcal{F}\mathbf{D}$.

It is a straightforward exercise to check that this definition is consistent and \mathcal{F} is indeed a 2-functor. Moreover, the diagram in the statement commutes because all 2-functors are the identity on objects, $\mathcal{F}(1_{\Phi})(f) = \Phi(f) = \mathcal{F}(\Phi)(f)$ and $\mathcal{F}(1_{\Phi})(h, k) = (\Phi(h), \Phi(k)) = \mathcal{F}(\Phi)(h, k)$. \square

3. Baues–Wirsching cohomology of categories

Definition 3.1. Recall from [2] that a *natural system* on \mathbf{C} is a functor $L: \mathcal{F}\mathbf{C} \rightarrow \mathbf{Ab}$. The *Baues–Wirsching complex* $F^*(\mathbf{C}, L)$ of a small category \mathbf{C} with coefficients in a natural system L on \mathbf{C} is a cochain complex of abelian groups concentrated in non-negative dimensions. In dimension n this complex is given by the following product indexed by all sequences of morphisms of length n in \mathbf{C} :

$$F^n(\mathbf{C}, L) = \prod_{\bullet \xleftarrow{\sigma_1} \dots \xleftarrow{\sigma_n} \bullet} L(\sigma_1 \cdots \sigma_n).$$

In this formula we assume that a sequence of length 0 is an object X in \mathbf{C} which we also identify with the identity morphism 1_X . The coordinate of $c \in F^n(\mathbf{C}, L)$ in $\bullet \xleftarrow{\sigma_1} \dots \xleftarrow{\sigma_n} \bullet$

will be denoted by $c(\sigma_1, \dots, \sigma_n)$. The differential d is defined as

$$\begin{aligned} d(c)(\sigma_1, \dots, \sigma_{n+1}) &= L(1, \sigma_1)c(\sigma_2, \dots, \sigma_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i c(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\ &\quad + (-1)^{n+1} L(\sigma_{n+1}, 1)c(\sigma_1, \dots, \sigma_n) \end{aligned}$$

over an n -cochain c for $n \geq 1$, and $d(c)(\sigma) = L(1, \sigma)c(X) - L(\sigma, 1)c(Y)$ for $n = 0$ and $\sigma: X \rightarrow Y$.

The cohomology of \mathbf{C} with coefficients in L is the cohomology of the complex $F^*(\mathbf{C}, L)$. It is denoted by $H^*(\mathbf{C}, L)$.

Baues and Wirsching noticed that H^* and F^* are functors in the category \mathbf{Nat} defined as follows:

objects: are pairs (\mathbf{C}, L) where L is a natural system on \mathbf{C} ,

morphisms: $(\Phi, t): (\mathbf{C}, L) \rightarrow (\mathbf{D}, M)$ are pairs given by a functor $\Phi: \mathbf{D} \rightarrow \mathbf{C}$ and a natural transformation $t: L\mathcal{F}(\Phi) \Rightarrow M$, and composition is given by the formula $(\Psi, s)(\Phi, t) = (\Phi\Psi, s \square (t1_{\mathcal{F}(\Psi)}))$.

Let **Cochain** be the category of cochain complexes of abelian groups and cochain homomorphisms. As a functor

$$F^*: \mathbf{Nat} \longrightarrow \mathbf{Cochain}$$

is defined as follows:

on objects: $F^*(\mathbf{C}, L)$ is the Baues–Wirsching complex;

on morphisms: $F^n(\Phi, t)(c)(\sigma_1, \dots, \sigma_n) = t_\sigma(c(\Phi(\sigma_1), \dots, \Phi(\sigma_n)))$, where $\sigma = \sigma_1 \cdots \sigma_n$

and

$$H^n = H^n F^*: \mathbf{Nat} \longrightarrow \mathbf{Ab}, \quad n \in \mathbb{Z}.$$

4. The Baues–Wirsching complex as a 2-functor

This section is the core of the paper. Its main goal is to extend F^* to a 2-functor from an adequate 2-category $\mathbf{Nat}_{\mathcal{F}}$ with the same objects as \mathbf{Nat} to the following 2-category

Cochain₂:

objects: cochain complexes of abelian groups;

1-morphisms: cochain homomorphisms, i.e., graded homomorphisms $p: A^* \rightarrow B^*$ of degree 0 such that $dp = pd$;

2-morphisms: $[h]: p \Rightarrow q$ are relative homotopy classes of homotopies between p and q , i.e., $[h]$ is represented by a degree -1 homomorphism $h: A^* \rightarrow B^*$ such that $dh + hd = -p + q$ and $[h] = [h']$ if there exists $r: A^* \rightarrow B^*$ of degree -2 such that $dr - rd = -h + h'$; vertical composition of 2-morphisms is given by $[h'] \square [h] = [h' + h]$, and the horizontal composition of $[h]$ and $[h']$ in the following diagram:

$$\begin{array}{ccccc} & & p & & p' \\ & \nearrow & & \searrow & \\ A^* & & & & B^* & & & C^* \\ & \searrow & & \nearrow & \\ & & q & & q' \end{array}$$

is $[h'] [h] = [h'p + q'h] = [p'h + h'q]$; one can use the degree -2 homomorphism $h'h: A^* \rightarrow C^*$ to check the last equality.

Notice that morphism categories in **Cochain₂** are in fact groupoids. Moreover, there is a unique 2-functor

$$v: \mathbf{Cochain} \longrightarrow \mathbf{Cochain}_2 \quad (4.A)$$

which is the identity on objects and 1-morphisms.

Let us define the 2-category **Nat _{\mathcal{F}}** :

(4.B) *objects:* are pairs (\mathbf{C}, L) where L is a natural system on \mathbf{C} ;

1-morphisms: $(\alpha, t): (\mathbf{C}, L) \rightarrow (\mathbf{D}, M)$ are pairs given by a natural transformation $\alpha: \Phi \Rightarrow \Psi$ between functors $\Phi, \Psi: \mathbf{D} \rightarrow \mathbf{C}$, or equivalently a morphism $\alpha: \mathbf{D} \rightarrow \mathbf{C}$ in **Cat _{\mathcal{F}}** (see (2.A)), and a natural transformation $t: L\mathcal{F}(\alpha) \Rightarrow M$, where \mathcal{F} is the 2-functor defined in Proposition 2.2, and composition is defined as $(\beta, s)(\alpha, t) = (\alpha\beta, s \square (t1_{\mathcal{F}(\beta)}))$;

2-morphisms: $(\varepsilon, \gamma): (\alpha, t) \Rightarrow (\beta, s)$ are 2-morphisms $(\varepsilon, \gamma): \alpha \Rightarrow \beta$ in **Cat _{\mathcal{F}}** such that $t = s \square (1_{L\mathcal{F}(\varepsilon, \gamma)})$, i.e., the following diagram of natural transformations commutes:

$$\begin{array}{ccc} L\mathcal{F}(\alpha) & & \\ \downarrow 1_{L\mathcal{F}(\varepsilon, \gamma)} & \searrow t & \\ L\mathcal{F}(\beta) & \nearrow s & M \end{array} \quad (4.C)$$

vertical and horizontal compositions of 2-morphisms in $\mathbf{Nat}_{\mathcal{F}}$ are defined as in $\mathbf{Cat}_{\mathcal{F}}$, i.e., $(\varepsilon', \gamma') \square (\varepsilon, \gamma) = (\varepsilon \square \varepsilon', \gamma' \square \gamma)$ and given a diagram in $\mathbf{Nat}_{\mathcal{F}}$

$$\begin{array}{ccccc} & (\alpha, t) & & (\alpha', t') & \\ & \curvearrowright & & \curvearrowright & \\ (\mathbf{C}, L) & & (\mathbf{D}, M) & & (\mathbf{E}, N) \\ & \curvearrowleft & & \curvearrowleft & \\ & (\beta, s) & & (\beta', s') & \end{array}$$

$\Downarrow (\varepsilon, \gamma) \qquad \Downarrow (\varepsilon', \gamma')$

the horizontal composition $(\varepsilon', \gamma')(\varepsilon, \gamma) = (\varepsilon \varepsilon', \gamma \gamma')$ in $\mathbf{Nat}_{\mathcal{F}}$ coincides with the horizontal composition of the following diagram in $\mathbf{Cat}_{\mathcal{F}}$:

$$\begin{array}{ccc} & \alpha & \\ & \curvearrowright & \\ \mathbf{C} & & \mathbf{D} \\ & \curvearrowleft & \\ & \beta & \end{array} \quad \begin{array}{ccc} & \alpha' & \\ & \curvearrowright & \\ \mathbf{D} & & \mathbf{E} \\ & \curvearrowleft & \\ & \beta' & \end{array}$$

$\Downarrow (\varepsilon, \gamma) \qquad \Downarrow (\varepsilon', \gamma')$

It is tedious but straightforward to check that $\mathbf{Nat}_{\mathcal{F}}$ is indeed a well-defined 2-category. Moreover, there is a unique 2-functor

$$j: \mathbf{Nat} \longrightarrow \mathbf{Nat}_{\mathcal{F}}$$

which is the identity on objects and sends a morphism (Φ, t) to $(1_{\Phi}, t)$. This makes sense because of the commutativity of the diagram in Proposition 2.2.

Theorem 4.1. *There is defined a 2-functor $F^*: \mathbf{Nat}_{\mathcal{F}} \rightarrow \mathbf{Cochain}_2$ fitting into a commutative diagram*

$$\begin{array}{ccc} \mathbf{Nat} & \xrightarrow{F^*} & \mathbf{Cochain} \\ \downarrow j & & \downarrow \iota \\ \mathbf{Nat}_{\mathcal{F}} & \xrightarrow{F^*} & \mathbf{Cochain}_2 \end{array}$$

Proof. The new 2-functor F^* is defined as follows. We use the notation in (2.A) and (4.B):

on objects: $F^*(\mathbf{C}, L)$ is the Baues–Wirsching complex;

on 1-morphisms:

$$F^*(\alpha, t)(c)(\sigma_1, \dots, \sigma_n) = t_{\sigma}(1_L \mathcal{F}(1_{\Phi}, \alpha))_{\sigma} c(\Phi(\sigma_1), \dots, \Phi(\sigma_n)).$$

Here $\sigma = \sigma_1 \cdots \sigma_n$ and the formula makes sense because

$$c(\Phi(\sigma_1), \dots, \Phi(\sigma_n)) \in L(\Phi(\sigma)) = (L \mathcal{F}(1_{\Phi}))(\sigma).$$

on 2-morphisms: $F^*(\varepsilon, \gamma) = [h_{(\varepsilon, \gamma)}]: F^*(\alpha, t) \rightarrow F^*(\beta, s)$ where for an $(n+1)$ -dimensional cochain c if $n > 0$ $h_{(\varepsilon, \gamma)}(c)$ is defined as

$$h_{(\varepsilon, \gamma)}(c)(\sigma_1, \dots, \sigma_n) = s_\sigma(1_{L\mathcal{F}}(1_\Gamma, \gamma \square \alpha))_\sigma \sum_{i=0}^n (-1)^i c(\Phi(\sigma_1), \dots, \Phi(\sigma_i), \varepsilon_{X_i}, \Gamma(\sigma_{i+1}), \dots, \Gamma(\sigma_n)),$$

where X_i is the source of σ_i and/or the target of σ_{i+1} . Notice that

$$c(\Phi(\sigma_1), \dots, \Phi(\sigma_i), \varepsilon_{X_i}, \Gamma(\sigma_{i+1}), \dots, \Gamma(\sigma_n)) \in (L\mathcal{F}(\varepsilon))(\sigma)$$

and if $n = 0$

$$h_{(\varepsilon, \gamma)}(c)(X) = s_{1_X}(1_{L\mathcal{F}}(1_\Gamma, \gamma \square \alpha))_{1_X} c(\varepsilon_X).$$

A tedious but straightforward computation shows that indeed

$$dh_{(\varepsilon, \gamma)} + h_{(\varepsilon, \gamma)}d = -F^*(\alpha, t) + F^*(\beta, s).$$

For this, essentially, one only needs to use the naturality property of natural transformations and the commutativity of (4.C).

It is easy to see that F^* preserves composition of 1-morphisms.

In order to check that F^* preserves vertical composition of 2-morphisms we consider a diagram in $\mathbf{Nat}_{\mathcal{F}}$

$$\begin{array}{ccc} & (\alpha, t) & \\ \swarrow & \Downarrow (\varepsilon, \gamma) & \searrow \\ (\mathbf{C}, L) & \xrightarrow{(\alpha', t')} & (\mathbf{D}, M) \\ \searrow & \Downarrow (\varepsilon', \gamma') & \swarrow \\ & (\beta, s) & \end{array}$$

where

$$\begin{array}{ccccc} \Psi & \xRightarrow{\gamma} & \Psi' & \xRightarrow{\gamma'} & \Upsilon \\ \uparrow \alpha & & \uparrow \alpha' & & \uparrow \beta \\ \Phi & \xleftarrow{\varepsilon} & \Phi' & \xleftarrow{\varepsilon'} & \Gamma \end{array}$$

is a commutative diagram of natural transformations between functors

$$\Phi, \Phi', \Psi, \Psi', \Gamma, \Upsilon: \mathbf{D} \rightarrow \mathbf{C}.$$

We define a degree -2 homomorphism

$$r_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}: F^*(\mathbf{C}, L) \longrightarrow F^*(\mathbf{D}, M)$$

in the following way, if c is an $(n+2)$ -cochain with $n > 0$ then

$$\begin{aligned} & r_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}(c)(\sigma_1, \dots, \sigma_n) \\ &= s_\sigma(1_L \mathcal{F}(1_\Gamma, \gamma' \square \gamma \square \alpha))_\sigma \sum_{i=0}^n \sum_{j=i}^n (-1)^{i+j} c(\Phi(\sigma_1), \dots, \\ & \quad \Phi(\sigma_i), \varepsilon_{X_i}, \Phi'(\sigma_{i+1}), \dots, \Phi'(\sigma_j), \varepsilon'_{X_j}, \Gamma(\sigma_{j+1}), \dots, \Gamma(\sigma_n)) \end{aligned}$$

and for $n = 0$

$$r_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}(c)(X) = s_{1_X}(1_L \mathcal{F}(1_\Gamma, \gamma' \square \gamma \square \alpha))_{1_X} c(\varepsilon_X, \varepsilon'_X).$$

It is hard but straightforward to check that

$$dr_{(\varepsilon', \gamma'); (\varepsilon, \gamma)} - r_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}d = -h_{(\varepsilon, \gamma)} - h_{(\varepsilon', \gamma')} + h_{(\varepsilon \square \varepsilon', \gamma' \square \gamma)}.$$

Therefore $F^*(\varepsilon', \gamma') \square F^*(\varepsilon, \gamma) = F^*(\varepsilon \square \varepsilon', \gamma' \square \gamma)$.

Let us see that F^* preserves horizontal composition of 2-morphisms. Consider a diagram in $\mathbf{Nat}_{\mathcal{F}}$

$$\begin{array}{ccccc} & & (\alpha, t) & & \\ & \curvearrowright & & \curvearrowright & \\ (\mathbf{C}, L) & & & & (\mathbf{D}, M) & & & & (\mathbf{E}, N) \\ & \Downarrow & (\varepsilon, \gamma) & \Downarrow & (\varepsilon', \gamma') & \\ & \curvearrowleft & & \curvearrowleft & & \\ & & (\beta, s) & & (\beta', s') & \end{array}$$

Here

$$\begin{array}{ccc} \psi & \xRightarrow{\gamma} & \gamma \\ \alpha \uparrow & & \uparrow \beta \\ \Phi & \xleftarrow{\varepsilon} & \Gamma \end{array} \quad \text{and} \quad \begin{array}{ccc} \psi' & \xRightarrow{\gamma'} & \gamma' \\ \alpha' \uparrow & & \uparrow \beta' \\ \Phi' & \xleftarrow{\varepsilon'} & \Gamma' \end{array}$$

are commutative diagrams of natural transformations between functors

$$\Phi, \Psi, \Gamma, \gamma: \mathbf{D} \rightarrow \mathbf{C} \quad \text{and} \quad \Phi', \Psi', \Gamma', \gamma': \mathbf{E} \rightarrow \mathbf{D}.$$

We define a degree -2 homomorphism

$$r'_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}: F^*(\mathbf{C}, L) \longrightarrow F^*(\mathbf{E}, N)$$

over an $(n + 2)$ -cochain c with $n > 0$ as

$$\begin{aligned} r'_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}(c)(\sigma_1, \dots, \sigma_n) \\ = s'_{\sigma}(s1_{\mathcal{F}(\beta')})_{\sigma}(1_L \mathcal{F}(1_{\Gamma'}(\gamma\gamma') \square(\alpha\alpha'))_{\sigma} \sum_{i=0}^n \sum_{j=i}^n (-1)^{i+j} c(\Phi\Phi'(\sigma_1), \dots, \\ \Phi\Phi'(\sigma_i), \Phi(\varepsilon'_{X_i}), \Phi\Gamma'(\sigma_{i+1}), \dots, \Phi\Gamma'(\sigma_j), \varepsilon_{\Gamma'(X_j)}, \Gamma\Gamma'(\sigma_{j+1}), \dots, \Gamma\Gamma'(\sigma_n)) \end{aligned}$$

and for $n = 0$

$$r'_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}(c)(X) = s'_{1_X}(s1_{\mathcal{F}(\beta')})_{1_X}(1_L \mathcal{F}(1_{\Gamma'}(\gamma\gamma') \square(\alpha\alpha'))_{1_X} c(\Phi(\varepsilon'_X), \varepsilon_{\Gamma'(X)}).$$

After a laborious computation one can check that

$$dr'_{(\varepsilon', \gamma'); (\varepsilon, \gamma)} - r'_{(\varepsilon', \gamma'); (\varepsilon, \gamma)}d = -h_{(\varepsilon', \gamma')}F^*(\alpha, t) - F^*(\beta', s')h_{(\varepsilon, \gamma)} + h_{(\varepsilon\varepsilon', \gamma\gamma')}.$$

Hence $F^*(\varepsilon', \gamma')F^*(\varepsilon, \gamma) = F^*(\varepsilon\varepsilon', \gamma\gamma')$.

The commutativity of the diagram in the statement follows easily from the commutativity of the diagram in Proposition 2.2. \square

The set $\pi_0\mathbf{C}$ of connected components of a small category \mathbf{C} is formed by equivalence classes $\{X\}$ of objects in \mathbf{C} . Two objects X, Y are equivalent $\{X\} = \{Y\}$ if there exists a sequence of (non-composable) morphisms in \mathbf{C} connecting them

$$X \rightarrow \bullet \leftarrow \dots \rightarrow \bullet \leftarrow Y.$$

This defines a product-preserving functor from small categories to sets

$$\pi_0: \mathbf{Cat} \longrightarrow \mathbf{Set}$$

with $\pi_0(\Phi)\{X\} = \{\Phi(X)\}$. Moreover, one can obtain an ordinary category \mathbf{M}^0 from a 2-category \mathbf{M} by taking π_0 on morphism categories, and also an ordinary functor $\rho^0: \mathbf{M}^0 \rightarrow \mathbf{N}^0$ from a 2-functor $\rho: \mathbf{M} \rightarrow \mathbf{N}$. If \mathbf{M} is a category regarded as a 2-category with only the trivial 2-morphisms then $\mathbf{M}^0 = \mathbf{M}$.

The homotopy category of cochain complexes $\mathbf{Cochain}/\simeq$ coincides with $\mathbf{Cochain}_2^0$ and the 2-functor $\iota: \mathbf{Cochain} \rightarrow \mathbf{Cochain}_2$ in (4.A) induces the natural projection $\iota^0: \mathbf{Cochain} \rightarrow \mathbf{Cochain}/\simeq$ onto the quotient category.

By Theorem 4.1 there is a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{Nat} & \xrightarrow{F^*} & \mathbf{Cochain} \\ j^0 \downarrow & & \downarrow \iota^0 \\ \mathbf{Nat}_{\mathcal{F}}^0 & \xrightarrow{(F^*)^0} & \mathbf{Cochain}/\simeq \end{array} \quad (4.D)$$

Proposition 4.2. *The functor $j^0: \mathbf{Nat} \rightarrow \mathbf{Nat}_{\mathcal{F}}^0$ is full.*

Proof. Let $(\alpha, t): (\mathbf{C}, L) \rightarrow (\mathbf{D}, M)$ be a morphism in $\mathbf{Nat}_{\mathcal{F}}$, as in (4.B). We consider the new morphism $(1_{\Phi}, t\Box(1_L\mathcal{F}(1_{\Phi}, \alpha))): (\mathbf{C}, L) \rightarrow (\mathbf{D}, M)$. Notice that $(1_{\Phi}, t\Box(1_L\mathcal{F}(1_{\Phi}, \alpha))) = j(\Phi, t\Box(1_L\mathcal{F}(1_{\Phi}, \alpha)))$. There is a 2-morphism in $\mathbf{Nat}_{\mathcal{F}}$

$$(1_{\Phi}, \alpha): (1_{\Phi}, t\Box(1_L\mathcal{F}(1_{\Phi}, \alpha))) \Rightarrow (\alpha, t).$$

Therefore

$$\{(\alpha, t)\} = \{j(\Phi, t\Box(1_L\mathcal{F}(1_{\Phi}, \alpha)))\} = j^0(\Phi, t\Box(1_L\mathcal{F}(1_{\Phi}, \alpha))). \quad \square$$

Proposition 4.2 shows that $\mathbf{Nat}_{\mathcal{F}}^0$ is a quotient category of \mathbf{Nat} , because j^0 is the identity on objects, and diagram (4.D) yields a factorization of the Baues–Wirsching cohomology functors through $\mathbf{Nat}_{\mathcal{F}}^0$.

Corollary 4.3. *The functors $H^n: \mathbf{Nat} \rightarrow \mathbf{Ab}$ ($n \in \mathbb{Z}$) factor through the quotient category $\mathbf{Nat}_{\mathcal{F}}^0$.*

5. (Co)localization and cohomology

In this section we prove two theorems, the localization and colocalization theorems, which allow to simplify computations in Baues–Wirsching cohomology of small categories when the coefficient natural systems satisfy some (co)locality conditions that will be made precise later.

Let \mathbf{D} be a reflective subcategory of \mathbf{C} , i.e., an adjoint pair $(\Phi$ left adjoint to $\Psi)$

$$\mathbf{C} \begin{matrix} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{matrix} \mathbf{D} \quad (5.A)$$

such that $\Phi\Psi = 1_{\mathbf{D}}$ and the counit is the identity natural transformation.

Remark 5.1. (1) A reflection can also be regarded as an idempotent functor $\Gamma: \mathbf{C} \rightarrow \mathbf{C}$ together with a natural transformation $\alpha: 1_{\mathbf{C}} \Rightarrow \Gamma$ such that $1_{\Gamma}\alpha = 1_{\Gamma} = \alpha 1_{\Gamma}$. This and the former situation are equivalent. On one hand, in this situation we can take \mathbf{D} to be the image of Γ , $\Psi: \mathbf{D} \rightarrow \mathbf{C}$ the (faithful) inclusion functor and $\Phi: \mathbf{C} \rightarrow \mathbf{D}$ the unique (full) functor such that $\Gamma = \Psi\Phi$, then we obtain a reflective subcategory as in (5.A) α being the unit of the adjunction. On the other hand, if we are as in (5.A) the functor $\Gamma = \Psi\Phi$ is idempotent and the unit of the adjunction $\alpha: 1 \Rightarrow \Gamma$ satisfies $1_{\Gamma}\alpha = 1_{\Gamma} = \alpha 1_{\Gamma}$.

(2) A related concept is that of a category of fractions. Given a set Σ of morphisms in \mathbf{C} the *category of fractions* $\mathbf{C}[\Sigma^{-1}]$ is a category together with a functor $\mathbf{C} \rightarrow \mathbf{C}[\Sigma^{-1}]$ which is universal among all functors which send morphisms in Σ to isomorphisms. Let \mathbf{C}_{Σ} be the full subcategory of \mathbf{C} formed by those objects X such that $\mathbf{C}(f, X)$ is a bijection for any $f \in \Sigma$ and $\Psi: \mathbf{C}_{\Sigma} \rightarrow \mathbf{C}$ the full inclusion. If there exists a functor $\Phi: \mathbf{C} \rightarrow \mathbf{C}_{\Sigma}$ as in (5.A) then the composite $\mathbf{C}_{\Sigma} \rightarrow \mathbf{C} \rightarrow \mathbf{C}[\Sigma^{-1}]$ is known to be an equivalence of categories. On

the other hand if we are in the situation of (5.A) and Σ is the set of morphisms f such that $\Phi(f)$ is an isomorphism then Ψ induces an equivalence between \mathbf{D} and \mathbf{C}_Σ .

Definition 5.2. Let \mathbf{D} be a reflective subcategory of \mathbf{C} as in (5.A). A natural system L on \mathbf{C} is said to be **D-local** if given $f: X \rightarrow Y$ in \mathbf{C} such that $\Phi(f)$ is an isomorphism then $L(1_X, f): L(1_X) \rightarrow L(f)$ is also an isomorphism.

The following proposition gives alternative characterizations of **D-local** natural systems.

Proposition 5.3. *Given a natural system L on \mathbf{C} the following conditions are equivalent:*

- (1) L is **D-local**,
- (2) L is naturally isomorphic to $M\mathcal{F}(\alpha)$ for some natural system M on \mathbf{C} ,
- (3) L is naturally isomorphic to $L\mathcal{F}(\alpha)$.

Here $\alpha: 1_{\mathbf{C}} \Rightarrow \Psi\Phi$ is the unit of the adjunction.

Proof. Let $f: X \rightarrow Y$ be a morphism in \mathbf{C} such that $\Phi(f)$ is an isomorphism. Then $M\mathcal{F}(\alpha)(1_X, f) = M(1_X, \Psi\Phi(f))$ is an isomorphism because $(1_X, \Psi\Phi(f))$ is an isomorphism in $\mathcal{F}\mathbf{C}$ with inverse $(1_X, \Psi(\Phi(f)^{-1}))$.

Suppose now that L is **D-local**. The natural transformation

$$1_{L\mathcal{F}(1_{\mathbf{C}}, \alpha)}: L = L\mathcal{F}(1_{\mathbf{C}}) \longrightarrow L\mathcal{F}(\alpha)$$

is a natural isomorphism since given an object f in $\mathcal{F}\mathbf{C}$, which is a morphism $f: X \rightarrow Y$ in \mathbf{C} , we have that $(1_{L\mathcal{F}(1_{\mathbf{C}}, \alpha)})_f = L(1_X, \alpha_Y)$ and $\Phi(\alpha_Y) = 1_{\Phi(Y)}$. For this last equality we use that Φ is left-adjoint to Ψ , α is the unit, and $\Phi\Psi = 1$. \square

Theorem 5.4. *Let \mathbf{D} be a reflective subcategory of \mathbf{C} as in (5.A). If L is a **D-local** natural system on \mathbf{C} then Ψ induces isomorphisms ($n \in \mathbb{Z}$)*

$$H^n(\mathbf{C}, L) \xrightarrow{\sim} H^n(\mathbf{D}, L\mathcal{F}(\Psi)).$$

Proof. The morphism in the statement is $H^n(\Psi, 1_{L\mathcal{F}(\Psi)})$. If $\alpha: 1_{\mathbf{C}} \Rightarrow \Psi\Phi$ is the unit we have that $\alpha\alpha = \alpha$, see Remark 5.1(1) and the beginning of the proof of the following lemma. Hence by Proposition 5.3 we can suppose without loss of generality that $L = L\mathcal{F}(\alpha)$. Consider the natural transformation

$$1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})}: L\mathcal{F}(\Psi)\mathcal{F}(\Phi) = L\mathcal{F}(\Psi\Phi) = L\mathcal{F}(1_{\Psi\Phi}) \Rightarrow L\mathcal{F}(\alpha) = L.$$

We claim that $H^n(1_\Phi, 1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})})$ is the inverse of

$$H^n(\Psi, 1_{L\mathcal{F}(\Psi)}) = H^n(1_\Psi, 1_{L\mathcal{F}(1_\Psi)}).$$

For this equality we use the commutativity of the diagram in Theorem 4.1. On one hand we have the following equalities in $\mathbf{Nat}_{\mathcal{F}}$

$$\begin{aligned}
 (1_{\Psi}, 1_{L\mathcal{F}(1_{\Psi})})(1_{\Phi}, 1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})}) &= (1_{\Phi}1_{\Psi}, 1_{L\mathcal{F}(1_{\Psi})}\square(1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})}1_{\mathcal{F}(1_{\Psi})})) \\
 &= (1_{\Phi\Psi}, 1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})}\mathcal{F}(1_{\Psi}, 1_{\Psi})) \\
 &= (1_{\mathbf{D}}, 1_{L\mathcal{F}(\alpha 1_{\Psi}, 1_{\Psi\Phi}1_{\Psi}))} \\
 &= (1_{\mathbf{D}}, 1_{L\mathcal{F}(1_{\Psi}, 1_{\Psi\Phi\Psi}))} \\
 &= (1_{\mathbf{D}}, 1_{L\mathcal{F}(1_{\Psi}, 1_{\Psi}))} \\
 &= (1_{\mathbf{D}}, 1_{L1_{\mathcal{F}(1_{\Psi})}}) \\
 &= (1_{\mathbf{D}}, 1_{L\mathcal{F}(1_{\Psi})}).
 \end{aligned}$$

Here we use the equality $\alpha 1_{\Psi} = 1_{\Psi}$ which can be checked by using the equality $\Phi\Psi = 1_{\mathbf{D}}$ and elementary properties of adjoint functors. On the other hand

$$\begin{aligned}
 (1_{\Phi}, 1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})})(1_{\Psi}, 1_{L\mathcal{F}(1_{\Psi})}) &= (1_{\Psi}1_{\Phi}, (1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})}\square(1_{L\mathcal{F}(1_{\Psi})}1_{\mathcal{F}(1_{\Phi})})) \\
 &= (1_{\Psi\Phi}, 1_{L\mathcal{F}(\alpha, 1_{\Psi\Phi})}).
 \end{aligned}$$

Now the theorem follows from Corollary 4.3, Remark 5.1(1) and the following lemma. \square

Lemma 5.5. *Let $\Gamma: \mathbf{C} \rightarrow \mathbf{C}$ and $\alpha: 1_{\mathbf{C}} \Rightarrow \Gamma$ be as in Remark 5.1(1) and L any natural system on \mathbf{C} . Then there is a diagram in $\mathbf{Nat}_{\mathcal{F}}$ as follows:*

$$\begin{array}{ccc}
 & 1 & \\
 & \curvearrowright & \\
 (C, L\mathcal{F}(\alpha)) & \xrightarrow{(\alpha, 1)} & (C, L\mathcal{F}(\alpha)) \\
 & \curvearrowleft & \\
 & (1_{\Gamma}, 1_{L\mathcal{F}(\alpha, 1_{\Gamma})}) &
 \end{array}$$

$\Downarrow (1_{1_{\mathbf{C}}}, \alpha)$
 $\Uparrow (\alpha, 1_{\Gamma})$

Proof. We have that $\alpha\alpha = (\alpha 1_{\Gamma})\square(1_{1_{\mathbf{C}}}\alpha) = 1_{\Gamma}\square\alpha = \alpha$. By using the 2-functor \mathcal{F} defined in Proposition 2.2 we see that $\mathcal{F}(\alpha)\mathcal{F}(1_{\Gamma}) = \mathcal{F}(\alpha 1_{\Gamma}) = \mathcal{F}(1_{\Gamma})$, $\mathcal{F}(\alpha, 1_{\Gamma}): \mathcal{F}(1_{\Gamma}) \rightarrow \mathcal{F}(\alpha)$, and $\mathcal{F}(\alpha)\mathcal{F}(\alpha) = \mathcal{F}(\alpha\alpha) = \mathcal{F}(\alpha)$. Therefore the 1-morphisms in the diagram are well defined. Moreover, the 2-morphisms are also well-defined because $1_{L\mathcal{F}(\alpha)} = 1_L 1_{\mathcal{F}(\alpha)}$, $1_{\mathcal{F}(\alpha)} = \mathcal{F}(1_{1_{\mathbf{C}}}, 1_{\Gamma})$ and

$$\mathcal{F}(1_{1_{\mathbf{C}}}, 1_{\Gamma})\mathcal{F}(1_{1_{\mathbf{C}}}, \alpha) = \mathcal{F}(1_{1_{\mathbf{C}}}1_{1_{\mathbf{C}}}, 1_{\Gamma}\alpha) = \mathcal{F}(1_{1_{\mathbf{C}}}, 1_{\Gamma}),$$

$$\mathcal{F}(1_{1_{\mathbf{C}}}, 1_{\Gamma})\mathcal{F}(\alpha, 1_{\Gamma}) = \mathcal{F}(1_{1_{\mathbf{C}}}\alpha, 1_{\Gamma}1_{\Gamma}) = \mathcal{F}(\alpha, 1_{\Gamma}). \quad \square$$

Example 5.6. Here we exhibit some examples of the usefulness of Theorem 5.4 as a computational tool in cohomology of categories. We recall from [2] that any functor $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Ab}$ yields a natural system on \mathbf{C} through the natural functor $\mathcal{F}\mathbf{C} \rightarrow \mathbf{C}^{\text{op}} \times \mathbf{C}$ which sends an object f , which is a morphism $f: X \rightarrow Y$ in \mathbf{C} , to the pair (X, Y) .

(1) In his study of the 2-dimensional cohomology groups of the category \mathbf{Ab}_0 of finitely generated abelian groups Hartl [6] computed

$$H^2(\mathbf{Ab}_0, \text{Ext}_{\mathbb{Z}}^1(-, - \otimes \mathbb{Z}/2)) \simeq \mathbb{Z}/2. \quad (5.B)$$

The generator is the characteristic class associated to the stable homotopy category of compact Moore spaces (see [2, 3.2]). Let $\Psi: \mathbf{vect}(\mathbb{Z}/2) \rightarrow \mathbf{Ab}_0$ be the inclusion of the full subcategory of finite-dimensional $\mathbb{Z}/2$ -vector spaces, whose left-adjoint is $\Phi = - \otimes \mathbb{Z}/2$. Clearly $\text{Ext}_{\mathbb{Z}}^1(-, - \otimes \mathbb{Z}/2)$ is $\mathbf{vect}(\mathbb{Z}/2)$ -local. Moreover, there is a natural isomorphism

$$\text{Ext}_{\mathbb{Z}}^1(-, - \otimes \mathbb{Z}/2) \mathcal{F}(\Psi) \simeq \text{Hom}_{\mathbb{Z}/2}$$

induced by the change of rings spectral sequence associated to the natural projection $\mathbb{Z} \rightarrow \mathbb{Z}/2$ (see [9, 10.2(c)]). Therefore by Theorem 5.4

$$H^n(\mathbf{Ab}_0, \text{Ext}_{\mathbb{Z}}^1(-, - \otimes \mathbb{Z}/2)) \simeq H^n(\mathbf{vect}(\mathbb{Z}/2), \text{Hom}_{\mathbb{Z}/2}), \quad n \in \mathbb{Z}.$$

Moreover, by Theorem A and Corollary 3.11 in [7] this is the Mac Lane cohomology of $\mathbb{Z}/2$,

$$H^n(\mathbf{vect}(\mathbb{Z}/2), \text{Hom}_{\mathbb{Z}/2}) \simeq H_{\text{ML}}^n(\mathbb{Z}/2, \mathbb{Z}/2) \simeq \begin{cases} \mathbb{Z}/2, & n \text{ even}; \\ 0, & n \text{ odd}; \end{cases}$$

which was computed in [4]. Therefore we recover the isomorphism (5.B) from this complete calculation of $H^*(\mathbf{Ab}_0, \text{Ext}_{\mathbb{Z}}^1(-, - \otimes \mathbb{Z}/2))$.

(2) More generally by Theorems A and B in [7] for any prime ℓ there is a spectral sequence converging to $H_{\text{ML}}^{n+m}(\mathbb{Z}, \mathbb{Z}/\ell)$ with

$$E_2^{nm} = H^n(\mathbf{Ab}_0, \text{Ext}_{\mathbb{Z}}^m(-, - \otimes \mathbb{Z}/\ell)).$$

Let $\Psi: \mathbf{vect}(\mathbb{Z}/\ell) \rightarrow \mathbf{Ab}_0$ be the full inclusion of the category of finite-dimensional \mathbb{Z}/ℓ -vector spaces, whose left-adjoint is $\Phi = - \otimes \mathbb{Z}/\ell$. It is obvious that $\text{Ext}_{\mathbb{Z}}^m(-, - \otimes \mathbb{Z}/\ell)$ is $\mathbf{vect}(\mathbb{Z}/\ell)$ -local for all $m \geq 0$. Moreover

$$\text{Ext}_{\mathbb{Z}}^m(-, - \otimes \mathbb{Z}/\ell) \mathcal{F}(\Psi) \simeq \begin{cases} \text{Hom}_{\mathbb{Z}/\ell} & \text{for } m = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for $m = 0, 1$

$$H^n(\mathbf{Ab}_0, \text{Ext}_{\mathbb{Z}}^m(-, - \otimes \mathbb{Z}/\ell)) \simeq H^n(\mathbf{vect}(\mathbb{Z}/\ell), \text{Hom}_{\mathbb{Z}/\ell}) \simeq H_{\text{ML}}^n(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$$

and zero for $m \neq 0, 1$. Therefore, we recover from the previous spectral sequence a long exact sequence ($n \in \mathbb{Z}$)

$$\cdots \rightarrow H_{\text{ML}}^{n-2}(\mathbb{Z}/\ell, \mathbb{Z}/\ell) \rightarrow H_{\text{ML}}^n(\mathbb{Z}/\ell, \mathbb{Z}/\ell) \rightarrow H_{\text{ML}}^n(\mathbb{Z}, \mathbb{Z}/\ell) \rightarrow H_{\text{ML}}^{n-1}(\mathbb{Z}/\ell, \mathbb{Z}/\ell) \rightarrow \cdots$$

which was used in [4] to compute the Mac Lane cohomology $H_{\text{ML}}^*(\mathbb{Z}, \mathbb{Z}/\ell)$.

Let \mathbf{D} be now a coreflective subcategory of \mathbf{C} , i.e., an adjoint pair $(\Phi \text{ right adjoint to } \Psi)$

$$\mathbf{C} \begin{matrix} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{matrix} \mathbf{D} \quad (5.C)$$

such that $\Phi\Psi = 1_{\mathbf{D}}$ and the unit is the identity natural transformation.

Dually to Remark 5.1 we have the following observations.

Remark 5.7. (1) A coreflection can also be regarded as an idempotent functor $\Gamma: \mathbf{C} \rightarrow \mathbf{C}$ together with a natural transformation $\alpha: \Gamma \Rightarrow 1_{\mathbf{C}}$ such that $1_{\Gamma}\alpha = 1_{\Gamma} = \alpha 1_{\Gamma}$. This is equivalent to the situation in (5.C). In fact (5.C) yields the structure described here by taking $\Gamma = \Psi\Phi$ and α the counit. Moreover, in the situation of this example we obtain a coreflection as in (5.C) by taking \mathbf{D} as the image of Γ , $\Psi: \mathbf{D} \rightarrow \mathbf{C}$ the inclusion and $\Phi: \mathbf{C} \rightarrow \mathbf{D}$ the unique functor such that $\Gamma = \Psi\Phi$.

(2) Given a set Σ of morphisms in \mathbf{C} we define $_{\Sigma}\mathbf{C}$ to be the full subcategory of \mathbf{C} formed by those objects X such that $\mathbf{C}(X, f)$ is a bijection for any $f \in \Sigma$. Let $\Psi: _{\Sigma}\mathbf{C} \rightarrow \mathbf{C}$ be the full inclusion. If there exists a functor $\Phi: \mathbf{C} \rightarrow _{\Sigma}\mathbf{C}$ as in (5.C) then the composite $_{\Sigma}\mathbf{C} \rightarrow \mathbf{C} \rightarrow \mathbf{C}[\Sigma^{-1}]$ is an equivalence of categories. On the other hand if we are in the situation of (5.C) and Σ is the set of morphisms f such that $\Phi(f)$ is an isomorphism then Ψ induces an equivalence between \mathbf{D} and $_{\Sigma}\mathbf{C}$.

Definition 5.8. Let \mathbf{D} be a coreflective subcategory of \mathbf{C} as in (5.C). A natural system L on \mathbf{C} is said to be **\mathbf{D} -colocal** if given $f: X \rightarrow Y$ in \mathbf{C} such that $\Phi(f)$ is an isomorphism then $L(f, 1_Y): L(1_Y) \rightarrow L(f)$ is also an isomorphism.

Proposition 5.9. *Given a natural system L on \mathbf{C} the following conditions are equivalent:*

- (1) L is **\mathbf{D} -colocal**,
- (2) L is naturally isomorphic to $M\mathcal{F}(\alpha)$ for some natural system M on \mathbf{C} ,
- (3) L is naturally isomorphic to $L\mathcal{F}(\alpha)$.

Here $\alpha: \Psi\Phi \Rightarrow 1_{\mathbf{C}}$ is the counit of the adjunction.

Proof. Given a morphism $f: X \rightarrow Y$ in \mathbf{C} such that $\Phi(f)$ is an isomorphism, $M\mathcal{F}(\alpha)(f, 1_Y) = M(\Psi\Phi(f), 1_Y)$ is an isomorphism because $(\Psi\Phi(f), 1_Y)$ is an isomorphism in $\mathcal{F}\mathbf{C}$ with inverse $(\Psi(\Phi(f)^{-1}), 1_Y)$.

Conversely if L is **\mathbf{D} -colocal** one can readily check that the natural transformation

$$1_{L\mathcal{F}(\alpha, 1_{1_{\mathbf{C}}})}: L = L\mathcal{F}(1_{1_{\mathbf{C}}}) \longrightarrow L\mathcal{F}(\alpha)$$

is a natural isomorphism, compare the proof of Proposition 5.3. \square

Theorem 5.10. *Let \mathbf{D} be a coreflective subcategory of \mathbf{C} as in (5.C). If L is a **\mathbf{D} -colocal** natural system on \mathbf{C} , then Ψ induces isomorphisms $(n \in \mathbb{Z})$*

$$H^n(\mathbf{C}, L) \xrightarrow{\sim} H^n(\mathbf{D}, L\mathcal{F}(\Psi)).$$

The proof of this theorem is very similar to that of Theorem 5.4, for this reason we leave it to the reader. In this case one has to use Proposition 5.9, Remark 5.7(1) and the following lemma instead of Proposition 5.3, Remark 5.1(1) and Lemma 5.5.

Lemma 5.11. *Let $\Gamma: \mathbf{C} \rightarrow \mathbf{C}$ and $\alpha: \Gamma \Rightarrow 1_{\mathbf{C}}$ be as in Remark 5.7(1) and let L be any natural system on \mathbf{C} . Then there is a diagram in $\mathbf{Nat}_{\mathcal{F}}$ as follows:*

$$\begin{array}{ccc}
 & 1 & \\
 \swarrow & \Downarrow (\alpha, 1_{\mathbf{C}}) & \searrow \\
 (\mathbf{C}, L_{\mathcal{F}}(\alpha)) & \xrightarrow{(\alpha, 1)} & (\mathbf{C}, L_{\mathcal{F}}(\alpha)) \\
 \searrow & \Uparrow (1_{\Gamma}, \alpha) & \swarrow \\
 & (1_{\Gamma}, 1_{L_{\mathcal{F}}}(1_{\Gamma}, \alpha)) &
 \end{array}$$

The proof of this lemma is a mere verification as in Lemma 5.5.

Example 5.12. Here we combine the localization and colocalization theorems (Theorems 5.4 and 5.10) to reduce the computation of certain cohomology groups of a triangulated category (see [5, Chapter IV] for basic facts).

More precisely, let \mathbf{D} be a small triangulated category, whose translation functor is denoted by Σ , equipped with a t -structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ and $\mathbf{A} = \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$ its core. We write $t_{\leq 0}: \mathbf{D} \rightarrow \mathbf{D}^{\leq 0}$ and $t_{\geq 0}: \mathbf{D} \rightarrow \mathbf{D}^{\geq 0}$ for the right and left adjoints of the corresponding full inclusions. Notice that the restriction of $t_{\geq 0}$ to $\mathbf{D}^{\leq 0}$, which is usually denoted by $H^0: \mathbf{D}^{\leq 0} \rightarrow \mathbf{A}$, is also a left-adjoint to the full inclusion.

We consider the natural system on \mathbf{D} given by the functor

$$\mathbf{D}(t_{\leq 0}, \Sigma^n t_{\geq 0}): \mathbf{D}^{\text{op}} \times \mathbf{D} \longrightarrow \mathbf{Ab}. \quad (5.D)$$

The restriction to $\mathbf{A}^{\text{op}} \times \mathbf{A}$ will be denoted by $\text{Ext}_{\mathbf{D}}^n$. Moreover, it coincides with the ordinary extension functor $\text{Ext}_{\mathbf{A}}^n$ in the abelian category \mathbf{A} when $\mathbf{D} = \mathcal{D}^b(\mathbf{A})$ is the bounded derived category of \mathbf{A} with the canonical t -structure.

By using elementary properties of triangulated categories and t -structures one can check that the natural system defined by (5.D) is $\mathbf{D}^{\leq 0}$ -colocal. Moreover, its restriction to $\mathbf{D}^{\leq 0}$, i.e., $\mathbf{D}^{\leq 0}(-, \Sigma^n H^0)$, is \mathbf{A} -local. Therefore

$$H^*(\mathbf{D}, \mathbf{D}(t_{\leq 0}, \Sigma^n t_{\geq 0})) \simeq H^*(\mathbf{D}^{\leq 0}, \mathbf{D}^{\leq 0}(-, \Sigma^n H^0)) \simeq H^*(\mathbf{A}, \text{Ext}_{\mathbf{D}}^n).$$

This and similar examples are relevant for the approach to homotopy theory given by the tower of categories in [1]. An explicit application can be found in [11].

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